9.6 Ratio and Root Tests

THEOREM 9.17 Ratio Test

Let Σa_n be a series with nonzero terms.

1. Σa_n converges absolutely if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. Σa_n diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Proof To prove Property 1, assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

and choose R such that $0 \le r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $|a_{n+1}/a_n| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$\begin{aligned}
|a_{N+1}| &< |a_N|R \\
|a_{N+2}| &< |a_{N+1}|R < |a_N|R^2 \\
|a_{N+3}| &< |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3 \\
&\vdots
\end{aligned}$$

The geometric series $\Sigma |a_N|R^n = |a_N|R + |a_N|R^2 + \cdots + |a_N|R^n + \cdots$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + \cdots + |a_{N+n}| + \cdots$$

also converges. This in turn implies that the series $\Sigma |a_n|$ converges, because discarding a finite number of terms $(n = N - 1)$ does not affect convergence. Consequently, by Theorem 9.16, the series Σa_n converges absolutely. The proof of Property 2 is similar and is left as an exercise (see Exercise 98).

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

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3. The Ratio Test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

$$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$$

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3. The Ratio Test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

THEOREM 9.18 Root Test

Let Σa_n be a series.

1. Σa_n converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. Σa_n diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

$$\sum_{n=1}^{\infty} 3^n e^{-n}$$

Note that this limit is not as easily evaluated as the limit obtained by the Root Test

THEOREM 9.18 Root Test

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3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Use the root test to determine whether the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{5} + \frac{1}{n}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\tan^{-1} n\right)^n$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{2^{(n^2)}}$

THEOREM 9.18 Root Test

Let $\Sigma\, a_n$ be a series.

1. $\Sigma\, a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\Sigma\, a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

By the way, it's important to remember that the root test is inconclusive when $r = 1$. Indeed, all of the following series have $r = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} 1 \qquad \sum_{n=1}^{\infty} n^3$$

The first series converges, while the remaining three all diverge. The only thing these series have in common is that none of them are exponential, and are therefore not susceptible to analysis by the root test.